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TWO EXTREMAL PROBLEMS

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ABSTRACT. The class H consists of all functions f which are analytic in the disk D and satisfy

$$\iint_D |f(z)|\, dx\, dy < \infty.$$

Linear extremal problems for functionals of the type $\Lambda(f) = \iint_D f(z)\phi(z)\, dx\, dy$, $\phi \in L^\infty(D)$, $f \in H$, are studied.

1. **Introduction.** We discuss two specific linear extremal problems, formulated on spaces of analytic functions in the disc.

The first is an extremal problem in H_∞, the space of bounded analytic functions defined on the open unit disc. There is a well-developed and useful theory which encompasses extremal problems in all the Hardy spaces H_p, $1 \leqslant p \leqslant \infty$. In its most general form, this theory is due to Havinson, Rogosinski and Shapiro [2], [5], and relies heavily on the notion of duality and the factorization theorems for functions in these classes. The theory is most complete in the case that the functionals in question are of the form

$$\Lambda(f) = \int_{|z|=1} f(z)k(z)\, dz, \qquad f \in H_p, \tag{1.1}$$

where $k(z)$ is a rational function which is continuous on the unit circle [2], [5]. Our first example is not of this form. However, it is found that complete information may be obtained by representing the functional as in (1.1) but with the integral instead taken over the interval $(-1, 1)$.

The second problem has been suggested by Professors Kurt Strebel and Edgar Reich. It arises as a special case in connection with the theory of quasiconformal mappings and in particular connection with a remarkable theorem of Hamilton [1]. We state here a special case of this theorem.

THEOREM (HAMILTON). *Let F be a quasiconformal map of the unit disc onto itself. Let Q_F be the set of all such quasiconformal maps G which also sat-*

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isfy $G(e^{i\theta}) = F(e^{i\theta})$, $0 \leq \theta \leq 2\pi$. *Let* κ_G *be the complex dilatation of* G *in* Q_F, *and let* F^* *be a member of* Q_F *which satisfies*

(1.2)
$$\|\kappa_{F^*}\|_\infty \leq \|\kappa_G\|_\infty$$

for all G *in* Q_F. *Denote by* H *the Banach space of functions* f *which are analytic in the open unit disc* D *and which satisfy*

(1.3)
$$\|f\| = \iint_D |f| \, dx \, dy < \infty.$$

Then we have

(1.4)
$$\sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \iint_D \kappa_{F^*}(z) f(z) \, dx \, dy \right| = \|\kappa_{F^*}\|_\infty .$$

In the context of the theorem above, F^* is called an extremal quasiconformal mapping of the disc onto itself.

Let $\kappa \in L^\infty(D)$ and suppose $\|\kappa\|_\infty < 1$. For such a function κ, denote by Λ_κ the functional,

(1.5)
$$\Lambda_\kappa(f) = \iint_D f(z)\kappa(z) \, dx \, dy, \qquad f \in H.$$

Λ_κ is an element of the dual space H^* of H. It has a norm in that space

(1.6)
$$\|\Lambda_\kappa\| = \sup_{\substack{f \in H \\ \|f\| \leq 1}} \left| \iint_D \kappa(z) f(z) \, dx \, dy \right| .$$

Clearly $\|\Lambda_\kappa\| \leq \|\kappa\|_\infty$.

Strebel and Reich have demonstrated the converse to Hamilton's Theorem [4]. With the terminology above, we have

THEOREM (HAMILTON, REICH, STREBEL). *A function* κ *in* $L^\infty(D)$ *is the complex dilatation of an extremal quasiconformal mapping of the disc to itself if and only if*

(1.7)
$$\|\Lambda_\kappa\| = \|\kappa\|_\infty < 1.$$

In light of this result, it would be interesting and useful to characterize, in some external manner, those κ in $L^\infty(D)$ which satisfy (1.7). There is an immediate result given by duality. Denote by $H^\perp$ the (closed) subspace of $L^\infty(D)$ consisting of functions $k(z)$ which satisfy

(1.8)
$$\iint_D f(z)k(z) \, dx \, dy = 0,$$

for all $f \in H$.

By duality, we have for any $\kappa \in L^\infty$,

$$\|\Lambda_\kappa\| = \min_{k \in H^\perp} \|\kappa - k\|. \tag{1.9}$$

That the minimum is assumed in (1.9) follows from well-known principles. Thus, $\|\Lambda_\kappa\| = \|\kappa\|_\infty$ if and only if 0 is a best approximation to κ in $H^\perp$.

A little more can be said if there is an extremal function f^* for Λ_κ. Let $\kappa \in L^\infty(D)$ be given and let k^* be a best approximation from $H^\perp$. Let $f^* \in H$ and satisfy

$$\|f^*\| = 1, \qquad \Lambda_\kappa(f^*) = \|\Lambda_\kappa\|. \tag{1.10}$$

Then

$$\kappa(z) - k^*(z) = \|\kappa - k^*\|_\infty \overline{f^*(z)}/|f^*(z)| \tag{1.11}$$

a.e. in D. The details are well known. In fact, a necessary and sufficient condition for f^* in H with $\|f^*\| = 1$ to be extremal, is that (1.11) hold for some k^* in $H^\perp$.

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2. **The first extremal problem.** The admissible functions are those in the closed unit ball of H_∞; that is, the set of all analytic functions f in the disc D satisfying

$$\|f\|_\infty = \sup_{|z|<1} |f(z)| \leq 1. \tag{2.1}$$

For z in D, define

$$\Phi(z) = \begin{cases} 1, & \operatorname{Im} z > 0, \\ -1, & \operatorname{Im} z < 0; \end{cases} \tag{2.2}$$

define the functional in H_∞^*, by

$$\Lambda_\Phi(f) = \iint_D f(z)\Phi(z)\,dx\,dy, \qquad f \in H_\infty. \tag{2.3}$$

LEMMA 2.1. *Let $f \in H_\infty$. We have*

$$\Lambda_\Phi(f) = i\int_{-1}^{1} (f(x) - f(0))\left(\frac{1}{x} - x\right) dx. \tag{2.4}$$

PROOF. Let D^+ be the upper half of the unit disc and $\Psi(z) = \Phi(z) + 1$, in D. Then

$$\Lambda_\Phi(f) = \iint_{D^+} \Psi(z)(f(z) - f(0))\,dx\,dy$$

$$= \frac{1}{i}\int_{\partial D^+} \bar{z}(f(z) - f(0))\,dz = \frac{1}{i}\int_{-1}^{1} \left(z - \frac{1}{z}\right)(f(z) - f(0))\,dz.$$

We have the following result.

THEOREM 2.1. $\max_{f \in H_\infty;\, \|f\| \leq 1} |\Lambda_\Phi(f)| = \Lambda_\Phi(-iz)$. *If* $\Lambda_\Phi(f) = \Lambda_\Phi(-iz)$ *and* $\|f\|_\infty = 1$, *then* $f(z) = -iz$.

PROOF. An extremal function f^* exists, satisfying $\Lambda_\Phi(f^*) = \|\Lambda_\Phi\|$, $\|f^*\|_\infty = 1$. The function $f_0^*(z) = (f^*(z) - f^*(-z))/2$ is also extremal and satisfies $\Lambda_\Phi(f_0^*) = \|\Lambda_\Phi\|$, $\|f_0^*\| \leq 1$. Note $f^* = f_0^* + f_e^*$ where $f_e^*(z) = (f^*(z) + f^*(-z))/2$.

By Lemma 2.1 and the conclusion from Schwarz's lemma that the extremal function f_0^* must satisfy $|f_0^*(z)| \leq |z|$, $|z| < 1$, we deduce that $f_0^*(z) = -iz$. Since for any even $f_e^*(z)$, $\Lambda_\Phi(f_e^*) = 0$, and since the H_∞ norm of $-iz + f_e^*$ will be greater than 1 unless $f_e^* \equiv 0$, we must have $f_e^* \equiv 0$; $f^*(z) = -iz$. $\square$

3. **Extremal problems in H.** As in §1, H is the set of all functions which are analytic in $|z| < 1$ and satisfy

$$\|f\| = \iint_D |f(z)|\,dx\,dy < \infty. \tag{3.1}$$

For $\phi \in L^\infty(D)$ define

$$\Lambda_\phi(f) = \iint_D f(z)\phi(z)\,dx\,dy, \tag{3.2}$$

$f \in H$. All elements of H^* (the dual space of H) are of the form (3.2), and

$$\|\Lambda_\phi\| = \sup_{\substack{\|f\| \leq 1\\ f \in H}} |\Lambda_\phi(f)|.$$

If Λ_ϕ is a functional in H^*, then a sequence $\langle f_n \rangle$ in H is said to be an extremal sequence for Λ_ϕ if $\|f_n\| = 1$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} |\Lambda_\phi(f_n)| = \|\Lambda_\phi\|$. A function f^* in H, of norm 1, is said to be an extremal function for Λ_ϕ if $|\Lambda_\phi(f^*)| = \|\Lambda_\phi\|$.

The extremal problems here are difficult in several respects. The dual problem is complicated by the necessity of finding the best uniform approximation over a *region.* Moreover, extremal functions clearly do not exist generally, although the unit ball in H is still a normal family of functions. In fact, for a large class of functions ϕ, an extremal sequence will necessarily vanish in the interior of the disc. To see this, take $\langle f_n \rangle$ in H satisfying $\|f_n\| = 1$, $n = 1, 2, \ldots,$

$\lim_{n\to\infty} f_n(z) = 0$, $|z| < 1$. There is a ϕ in L^∞ with the following properties

(1) ϕ continuous in $|z| < 1$,

(2) $|\phi(z)| < 1$, $|z| < 1$,

(3) $\overline{\lim}_{n\to\infty} |\iint_D f_n \phi \, dx \, dy| = 1$.

There can be no extremal function for Λ_ϕ and any extremal sequence must vanish in the disc, since $\|\Lambda_\phi\| = 1$.

We give two propositions which somewhat reflect the possibilities.

PROPOSITION 3.1. *Let $\langle f_n \rangle$ be a sequence in H which converges to f^* uniformly on compact subsets of D. Let $\|f_n\| = 1, n = 1, 2, \ldots$. Then $\|f^*\| \leqslant 1$ and*

$$\lim_{n\to\infty} \|f_n - f^*\| = 1 - \|f^*\|. \tag{3.3}$$

In Proposition 3.1, $\|f^*\| = 1$ implies that $\|f_n - f^*\| \longrightarrow 0$, whence $\Lambda_\phi(f_n)$ converges to $\Lambda_\phi(f^*)$ for any ϕ in $L^\infty(D)$.

PROOF. It suffices to show that for any subsequence (also denoted by $\langle f_n \rangle$), we have

$$\overline{\lim_{n\to\infty}} \|f_n - f^*\| \leqslant 1 - \|f^*\|. \tag{3.4}$$

It is clear that $\|f^*\| \leqslant 1$.

For $0 < r < 1$, let

$$D_r = \{z\colon |z| < r\}, \qquad A_r = \{z\colon r < |z| < 1\}.$$

Let $\epsilon > 0$. Choose $0 < r < 1$ so that $\iint_{A_r} |f^*| \, dx \, dy < \epsilon$; choose N sufficiently large so that for $n \geqslant N$ we have $\iint_{D_r} |f_n - f^*| \, dx \, dy < \epsilon$. Then for $n \geqslant N$,

$$\begin{aligned}\iint_D |f_n - f^*| \, dx \, dy &\leqslant \epsilon + \iint_{A_r} |f_n - f^*| \, dx \, dy \\ &\leqslant 2\epsilon + \iint_{A_r} |f_n| \, dx \, dy = 2\epsilon + 1 - \iint_{D_r} |f_n| \, dx \, dy.\end{aligned}$$

Hence

$$\begin{aligned}\overline{\lim_{n\to\infty}} \iint_D |f_n - f^*| \, dx \, dy &\leqslant 2\epsilon + 1 - \iint_{D_r} |f^*| \, dx \, dy \\ &\leqslant 3\epsilon + 1 - \iint_D |f^*| \, dx \, dy.\end{aligned}$$

PROPOSITION 3.2. *Let $\phi \in L^\infty(D)$ and Λ_ϕ the corresponding element in H^*. Then either*:

(1) *there exists an extremal function f^* for Λ_ϕ; or*

(2) *there exists an extremal sequence $\langle f_n \rangle$ for Λ_ϕ which converges to zero uniformly on compact subsets of D.*

PROOF. Let $\langle f_n \rangle$ be an extremal sequence and suppose $\langle f_n \rangle$ converges uniformly on compact subsets of D to f^*. If $f^* = 0$ or if $\|f^*\| = 1$, we are done (by Proposition 3.1). Otherwise, let

$$g^* = \|f^*\|^{-1} f^*, \quad g_n = \|f_n - f^*\|^{-1}(f_n - f^*).$$

Then

$$\begin{aligned}\|\Lambda_\phi\| &= \lim_{n\to\infty} |\Lambda_\phi(f_n)| \\ &\leqslant \left(\lim_{n\to\infty} \|f_n - f^*\|\right)\left(\overline{\lim_{n\to\infty}} |\Lambda_\phi(g_n)|\right) + \|f^*\|\,|\Lambda_\phi(g^*)| \\ &= (1 - \|f^*\|)(\overline{\lim}\,|\Lambda_\phi(g_n)| + \|f^*\|\,|\Lambda_\phi(g^*)| \leqslant \|\Lambda_\phi\|.\end{aligned}$$

Note $\langle g_n \rangle$ converges to 0 uniformly on compact subsets since $\lim_{n\to\infty} \|f_n - f^*\| \neq 0$. Hence g^* is extremal and $\langle g_n \rangle$ is a vanishing extremal sequence. □

In [3], [4] there are examples indicating that the above possibilities are *not* mutually exclusive. We may have both an extremal function and a vanishing extremal sequence for the same functional.

4. The functional of Reich and Strebel.(3) Take Φ as in §2, and Λ_Φ as in §3. We show that Λ_Φ possesses an extremal function. This will follow from the next result which shows the effect of Λ_Φ when acting on a vanishing sequence in the unit ball of H.

THEOREM 4.1. *Let $\langle f_n \rangle$ be a sequence in H satisfying $\|f_n\| \leqslant 1$, $n = 1, 2, 3, \ldots$, $\lim_{n\to\infty} f_n(z) = 0$ for all $|z| < 1$. Then,*

$$\overline{\lim_{n\to\infty}}\, |\Lambda_\Phi(f_n)| \leqslant 2/\pi. \tag{4.1}$$

If c_n satisfy $|c_n| = \|(1+z)^{1/n-2}\|^{-1}$, and $\Lambda_\Phi(c_n(1+z)^{1/n-2}) > 0$, $n = 1, 2, \ldots$, then

$$\lim_{n\to\infty} \Lambda_\Phi(c_n(1+z)^{1/n-2}) = \frac{2}{\pi} \tag{4.2}$$

and $c_n(1+z)^{1/n-2} \longrightarrow 0$ as $n \longrightarrow \infty$ for all z, $|z| < 1$.

PROOF. (1) We may take a sequence $\langle f_n \rangle$ with the following properties:
(1) each f_n is an odd polynomial;
(2) $\Lambda_\Phi(f_n) > 0$;
(3) $\|f_n\| \leqslant 1, f_n(z) \longrightarrow 0$, $|z| < 1$.

(3) The fact that $\|\Lambda_\Phi\| < 1$ has been previously demonstrated by E. Reich and K. Strebel, and more recently by E. Reich [3b]. In the earlier work, quasiconformal mappings are employed and in the latter work an analytic proof is given which employs duality. The corresponding approximation problem is difficult. The techniques are ingenious and of interest in their own right.

In the present work, however, the existence of an extremal function and the methods employed are perhaps new.

We employ the representation (2.4) of §2, except that the path of integration is shifted. Specifically, for each θ, $-\pi/2 < -\rho \leq \theta \leq \rho < \pi/2$, we define a path γ_θ. The path γ_θ contains a segment $te^{i\theta} - 1$, $0 \leq t \leq \partial(\rho)$, and a segment from $\partial(\rho)e^{i\theta} - 1$ to 0; γ_θ shall connect -1 to 1 and be symmetric with respect to the origin. The quantity $\partial(\rho)$ is positive, and chosen appropriately.

The sequence f_n vanishes in the interior. Hence for $-\rho \leq \theta \leq \rho$ we have

$$\Lambda_\Phi(f_n) = 2i \int_{-1}^{-1+\partial(\rho)e^{i\theta}} f_n(z)\left(\frac{1}{z} - z\right) dz + \epsilon(n, \theta), \tag{4.3}$$

where $\lim_{n\to\infty} \epsilon(\theta, n) = 0$ uniformly for $-\rho \leq \theta \leq \rho$. We then have $\int_{-\rho}^{\rho} |\epsilon(\theta, n)|\, d\theta = \epsilon_1(n)$ approaches 0 as $n \longrightarrow \infty$. From (4.3) we have

$$\begin{aligned} \Lambda_\Phi(f_n) &= 4i\int_0^{\partial(\rho)} f_n(te^{i\theta} - 1)te^{2i\theta}\, dt \\ &\quad - 2i\int_0^{\partial(\rho)} f_n(te^{i\theta} - 1)\frac{t^2e^{3i\theta}}{te^{i\theta} - 1}\, dt + \epsilon(n, \theta). \end{aligned} \tag{4.4}$$

Take $\partial(\rho)$ to be less than 1/2, and integrate both sides of (4.4) with respect to θ, from $-\rho$ to ρ. We receive

$$\begin{aligned} 2\rho\Lambda_\Phi(f_n) &\leq 4\int_{-\rho}^{\rho}\int_0^{\partial(\rho)} |f_n(te^{i\theta} - 1)|\, t\, dt\, d\theta \\ &\quad + 2\partial(\rho)\int_{-\rho}^{\rho}\int_0^{\partial(\rho)} |f_n(te^{i\theta} - 1)|\, t\, dt\, d\theta + \epsilon_1(n). \end{aligned} \tag{4.5}$$

Thus,

$$2\rho\Lambda_\Phi(f_n) \leq (4 + 2\partial(\rho)) \iint_{S(\rho)} |f_n(z)|\, dx\, dy + \epsilon_1(n), \tag{4.6}$$

with $S(\rho) = \{te^{i\theta} - 1 \colon -\rho \leq \theta \leq \rho,\ 0 \leq t \leq \partial(\rho)\}$. Since f_n is odd, $\iint_{S(\rho)} |f_n(z)|\, dx\, dy \leq 1/2$. Hence $2\rho\Lambda_\Phi(f_n) \leq 2 + \partial(\rho) + \epsilon_1(n)$. Allow $n \longrightarrow \infty$, $\partial(\rho) \longrightarrow 0$, $\rho \longrightarrow \pi/2$ in that order. We see

$$\overline{\lim_{n\to\infty}}\, \Lambda_\Phi(f_n) \leq \frac{2}{\pi}.$$

To show (4.2) we first derive a lower bound for $n|c_n|$. Let

$$S = \{-1 + te^{i\theta} \colon 0 < t < 2, -\pi/2 < \theta < \pi/2\}.$$

Then $D \subset S$. Let $f_n(z) = c_n(1 + z)^{1/n-2}$. We have

$$1 = \iint_D |f_n(z)|\, dx\, dy < \iint_S |f_n(z)|\, dx\, dy = \pi|c_n|n \cdot 2^{1/n}$$

so

$$n|c_n| > 1/\pi 2^{1/n}. \tag{4.7}$$

Also, since $(1+z)^{-2}$ is not summable on the disc, $\lim_{n\to\infty}|c_n| = 0$.

Choose $\rho > 0$, $\partial > 0$ so that

$$\{-1 + te^{i\theta}: 0 < t < \partial, 0 < \theta < \rho\} \subset D.$$

Then

$$\begin{aligned}\Lambda_\Phi(c_n(1+z)^{1/n-2}) &= |c_n| \iint_{|z|<1} |\mathrm{Im}((1+z)^{1/n-2})|\,dx\,dy \\ &> 2|c_n| \int_0^\rho \int_0^\partial r^{1/n-1}\sin(2-1/n)\theta\,dr\,d\theta \\ &= 2|c_n|n(2)^{1/n}[1-\cos(2-1/n)\rho](2-1/n)^{-1}.\end{aligned}$$

Using (4.7), and letting $n \to \infty$, $\partial \to 0$ and $\rho \to \pi/2$, we see

$$\varliminf_{n\to\infty} \Lambda_\phi(c_n(1+z)^{1/n-2}) \geqslant \frac{2}{\pi}.$$

Since $c_n \to 0$, we have

$$\lim_{n\to\infty} \Lambda_\phi(c_n(1+z)^{1/n-2}) = \frac{2}{\pi}. \quad \square$$

By Theorem 4.1, and Proposition 3.2, we see that there is an extremal function for Λ_Φ. Indeed, $\Lambda_\Phi(-iz) = 4/3$ and $\|iz\| = 2\pi/3$. Hence $1 > \|\Lambda_\Phi\| \geqslant 2/\pi$. Condition (1.11) shows that $-iz$ is not extremal, and we will derive a better lower bound for $\|\Lambda_\Phi\|$ in the next section.

5. **A lower bound for $\|\Lambda_\Phi\|$.** Consider $f^*(z) = -iz/(1-z^2)$, $|z| < 1$. f^* is an element of H and by (2.4), $\Lambda_\Phi(f^*) = 2$. Also

$$\|f^*\| = \iint_D \left|\frac{z}{1+z^2}\right| dx\,dy.$$

Let $g(z) = \int_0^z \sqrt{t/(1+t^2)}\,dt$, for z in $Q^+ = \{re^{i\theta}: 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \pi/2\}$. Then $\|f^*\| = 4\iint_{Q^+} |g'(z)|^2\,dx\,dy$. $g(Q^+)$ is a Jordan region bounded by the segments $\gamma_1 = [0, g(1)]$, $\gamma_2 = [g(i), 0]$, and the curve

$$\gamma_3: t \to x(t) + iy(t) = g(e^{it}), \qquad 0 \leqslant t \leqslant \pi/2.$$

We have

$$\iint_{Q^+} |g'(z)|^2\,dx\,dy = \int_{\gamma=\gamma_1+\gamma_2+\gamma_3} x\,dy.$$

Let $I = \int_0^{\pi/2} \sqrt{\cos\theta}\,d\theta$. We have $g(1) = (\sqrt{2} - I/\sqrt{2})$, $g(i) = e^{3\pi i/4}I$. Hence

$\int_{\gamma_2} x\,dy = I^2/4$. On γ_3,

$$x(t) = g(1) + \sqrt{2}(\sqrt{\cos t} - 1)\ \frac{dy}{dt} = \sqrt{\frac{\cos t}{2}},$$

hence

$$\int_{\gamma_3} x\,dy = 1 + \left(\frac{g(1)}{\sqrt{2}} - 1\right) I.$$

Hence

$$\iint_Q |g'(z)|^2\,dx\,dy = 1 + \frac{I^2}{4} + \left(\frac{g(1)}{\sqrt{2}} - 1\right) I = 1 + \frac{I^2}{4} - \frac{I^2}{2} = 1 - \frac{I^2}{4}.$$

Thus

$$\iint_D |f^*(z)|\,dx\,dy = 4 - I^2$$

and

$$\|\Lambda_\Phi\| \geqslant 2/(4 - I^2) > .779 > 2/\pi,$$

the value of I being approximately 1.1981.

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